

EIGENVALUES OF HERMITIAN MATRICES AND EQUIVARIANT COHOMOLOGY OF GRASSMANNIANS

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ABSTRACT. The saturation theorem of [Knutson-Tao '99] concerns the nonvanishing of Littlewood-Richardson coefficients. In combination with work of [Klyachko '98], it implies [Horn '62]'s conjecture about eigenvalues of sums of Hermitian matrices. This eigenvalue problem has a generalization [Friedland '00] to *majorized* sums of Hermitian matrices.

We further illustrate the common features between these two eigenvalue problems and their connection to Schubert calculus of Grassmannians. Our main result gives a Schubert calculus interpretation of Friedland's problem, via *equivariant* cohomology of Grassmannians. In particular, we prove a saturation theorem for this setting. Our arguments employ the aforementioned work together with [Thomas-Yong '12].

1. INTRODUCTION AND THE MAIN RESULTS

1.1. Eigenvalue problems of A. Horn and of S. Friedland. A. Horn's eigenvalue problem, first solved over a dozen years ago, asks how imposing the condition $A + B = C$ on three $r \times r$ Hermitian matrices constrains their eigenvalues λ , μ , and ν , written as nondecreasing vectors of real numbers. A. Klyachko [Kl98] showed $(\lambda, \mu, \nu) \in \mathbb{R}^{3r}$ must satisfy certain linear inequalities. Moreover, he showed that these inequalities give at least an asymptotic solution to the problem of which Littlewood-Richardson coefficients $c_{\lambda, \mu}^{\nu}$ are nonzero. More precisely, suppose λ, μ, ν are partitions with at most r parts. Klyachko showed that if $c_{\lambda, \mu}^{\nu} \neq 0$, then $(\lambda, \mu, \nu) \in \mathbb{Z}_{\geq 0}^{3r}$ satisfies his inequalities; conversely, if $(\lambda, \mu, \nu) \in \mathbb{Z}_{\geq 0}^{3r}$ satisfy his inequalities then $c_{N\lambda, N\mu}^{N\nu} \neq 0$ for some $N \in \mathbb{N}$. (Here, $N\lambda$ is the partition with each part of λ stretched by a factor of N .)

To strengthen Klyachko's converse, A. Knutson-T. Tao [KnTa99] established the *saturation theorem*: $c_{\lambda, \mu}^{\nu} \neq 0$ if and only if $c_{N\lambda, N\mu}^{N\nu} \neq 0$. Combined with [Kl98], it follows that Klyachko's inequalities agree with the inequalities conjectured in [Ho62] for the above eigenvalue problem. See, e.g., [Fu00b] for a survey of these problems.

The Littlewood-Richardson coefficients are structure constants for multiplication of Schur polynomials. Therefore, they can be alternatively interpreted as tensor product multiplicities in the representation theory of GL_n , or as intersection multiplicities in the Schubert calculus of Grassmannians. Indeed, [KnTa99] adopts the former viewpoint, providing conjectural extensions to other Lie groups. Subsequent work includes [KaMi08, BeKu10, Ku10, Res10, Sa12]; see also the references therein.

The main goal of this paper is to provide further evidence of the naturality of the connection of Horn's problem to Schubert calculus. We demonstrate how the connection continues to an extension of this eigenvalue problem that we now recall. A Hermitian matrix M **majorizes** another Hermitian matrix M' if $M - M'$ is positive semidefinite (its eigenvalues are all nonnegative). In this case, we write $M \geq M'$. S. Friedland [Fr00]

considered the following question:

Which eigenvalues (λ, μ, ν) can occur if $A + B \geq C$?

His solution is in terms of linear inequalities, which includes Klyachko's inequalities, a trace inequality and some additional inequalities. Later, W. Fulton [Fu00a] proved the additional inequalities are unnecessary. See followup work by A. Buch [Bu06] and by C. Chindris [Ch06] (who extends the work of H. Derksen-J. Weyman [DeWe00]).

Our finding is that the solution to S. Friedland's problem also governs the *equivariant* Schubert calculus of Grassmannians. This parallels the Horn problem's connection to classical Schubert calculus, but separates the problem from the GL_n -representation theory perspective.

Let $C_{\lambda, \mu}^\nu$ be the equivariant Schubert structure coefficient (defined in Section 1.2). The analogy with the earlier results is illustrated by:

Theorem 1.1 (Equivariant saturation). $C_{\lambda, \mu}^\nu \neq 0$ if and only if $C_{N \cdot \lambda, N \cdot \mu}^{N \cdot \nu} \neq 0$ for any $N \in \mathbb{N}$.

When $|\lambda| + |\mu| = |\nu|$ then $C_{\lambda, \mu}^\nu = c_{\lambda, \mu}^\nu$. Hence Theorem 1.1 actually generalizes the saturation theorem. That said, our proofs rely on the classical Horn inequalities and so do not provide an independent proof of the earlier results. In addition, we use the recent combinatorial rule for $C_{\lambda, \mu}^\nu$ developed by H. Thomas and the third author [ThYo12].¹

1.2. Equivariant cohomology of Grassmannians. Let $\text{Gr}_r(\mathbb{C}^n)$ denote the Grassmannian of r -dimensional subspaces $V \subseteq \mathbb{C}^n$. This space comes with an action of the torus $T = (\mathbb{C}^*)^n$ (induced from the action of T on \mathbb{C}^n). Therefore, it makes sense to discuss the *equivariant cohomology ring* $H_T^* \text{Gr}_r(\mathbb{C}^n)$. This ring is an algebra over the polynomial ring $\mathbb{Z}[t_1, \dots, t_n]$. (A more complete exposition of equivariant cohomology may be found in, e.g., [Fu07].)

As a $\mathbb{Z}[t_1, \dots, t_n]$ -module, $H_T^* \text{Gr}_r(\mathbb{C}^n)$ has a basis of *Schubert classes*. To define these, fix the flag of subspaces $F_\bullet : 0 \subset F_1 \subset F_2 \subset \dots \subset F_n = \mathbb{C}^n$, where F_i is the span of the standard basis vectors $e_n, e_{n-1}, \dots, e_{n+1-i}$. For each Young diagram λ inside the $r \times (n-r)$ rectangle, which we denote Λ , there is a corresponding **Schubert variety**, defined by

$$X_\lambda := \{V \subseteq \mathbb{C}^n \mid \dim(V \cap F_{n-r+i-\lambda_i}) \geq i, \text{ for } 1 \leq i \leq r\}.$$

Since X_λ is invariant under the action of T , and has codimension $2|\lambda|$, it determines a class $[X_\lambda]$ in $H_T^{2|\lambda|} \text{Gr}_r(\mathbb{C}^n)$. As λ varies over all Young diagrams inside Λ , the classes $[X_\lambda]$ form a basis for $H_T^* \text{Gr}_r(\mathbb{C}^n)$ over $\mathbb{Z}[t_1, \dots, t_n]$. We can therefore expand any product in $H_T^* \text{Gr}_r(\mathbb{C}^n)$ as

$$[X_\lambda] \cdot [X_\mu] = \sum_{\nu \subseteq \Lambda} C_{\lambda, \mu}^\nu [X_\nu],$$

where the coefficients $C_{\lambda, \mu}^\nu \in \mathbb{Z}[t_1, \dots, t_n]$ are the **equivariant Schubert structure coefficients**. By homogeneity, $C_{\lambda, \mu}^\nu$ is a polynomial of degree $|\lambda| + |\mu| - |\nu|$. In particular, this coefficient is zero unless $|\lambda| + |\mu| \geq |\nu|$.

The polynomials $C_{\lambda, \mu}^\nu$ depend on the parameters r and n , but our notation drops this dependency, with the following justification. First, we already fixed r . Next, the standard embedding $\iota: \text{Gr}_r(\mathbb{C}^n) \hookrightarrow \text{Gr}_r(\mathbb{C}^{n+1})$ induces a map $\iota^*: H_T^* \text{Gr}_r(\mathbb{C}^{n+1}) \rightarrow H_T^* \text{Gr}_r(\mathbb{C}^n)$.

¹The easy direction of (equivariant) saturation, $C_{\lambda, \mu}^\nu \neq 0 \Rightarrow C_{N \cdot \lambda, N \cdot \mu}^{N \cdot \nu} \neq 0$, can be proved directly by using this rule (or others). However, as in the classical situation, it is the converse that is nonobvious.

Using superscripts to indicate where a subvariety lives, we have $\iota^{-1}X_\lambda^{(n+1)} = X_\lambda^{(n)}$, and therefore $\iota^*[X_\lambda^{(n+1)}] = [X_\lambda^{(n)}]$. Let us write H_T^* for the graded inverse limit of these equivariant cohomology rings, so it is an algebra over $\mathbb{Z}[t_1, t_2, \dots]$. Write $\hat{\sigma}_\lambda \in H_T^*$ for the stable limit of the Schubert classes $[X_\lambda^{(n)}]$. The same structure constants $C_{\lambda, \mu}^\nu$ describe $\hat{\sigma}_\lambda \cdot \hat{\sigma}_\mu$ in this limit, so we can work in that limit without reference to n .

1.3. Inequalities for $C_{\lambda, \mu}^\nu$ and for eigenvalues. We will deduce Theorem 1.1 from inequalities describing the nonvanishing of $C_{\lambda, \mu}^\nu$. Let $[r] := \{1, 2, \dots, r\}$. For any $I = \{i_1 < i_2 < \dots < i_d\} \subseteq [r]$ define the partition

$$\lambda(I) := (i_d - d \geq \dots \geq i_2 - 2 \geq i_1 - 1).$$

This defines a bijection between subsets of $[r]$ of cardinality d and partitions whose Young diagrams are contained in a $d \times (r - d)$ rectangle. The following theorem combines the main results of [KL98, KnTa99].

Theorem 1.2. ([KL98], [KnTa99]) *Let λ, μ, ν be partitions with at most r parts such that*

$$(1) \quad |\lambda| + |\mu| = |\nu|.$$

The following are equivalent:

- (i) $c_{\lambda, \mu}^\nu \neq 0$.
- (ii) *For every $d < r$, and every triple of subsets $I, J, K \subseteq [r]$ of cardinality d such that $c_{\lambda(I), \lambda(J)}^{\lambda(K)} \neq 0$, we have*

$$(2) \quad \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j \geq \sum_{k \in K} \nu_k.$$

- (iii) *There exist $r \times r$ Hermitian matrices A, B, C with eigenvalues λ, μ, ν such that $A + B = C$.*

We are now ready to state our main result, which is a generalization of Theorem 1.2.

Theorem 1.3. *Let λ, μ, ν be partitions with at most r parts such that*

$$(3) \quad |\lambda| + |\mu| \geq |\nu| \text{ and } \max\{\lambda_i, \mu_i\} \leq \nu_i \text{ for all } i \leq r.$$

The following are equivalent:

- (i) $C_{\lambda, \mu}^\nu \neq 0$.
- (ii) *For every $d < r$, and every triple of subsets $I, J, K \subseteq [r]$ of cardinality d such that $c_{\lambda(I), \lambda(J)}^{\lambda(K)} \neq 0$, we have*

$$\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j \geq \sum_{k \in K} \nu_k.$$

- (iii) *There exist $r \times r$ Hermitian matrices A, B, C with eigenvalues λ, μ, ν such that $A + B \geq C$.*

Theorem 1.3 asserts that the main recursive inequalities (ii) controlling nonvanishing of $C_{\lambda, \mu}^\nu$ are just Horn's inequalities (2). The only difference between the governing inequalities lies in (1) versus (3). Notice that the second condition in (3) is unnecessary in Theorem 1.2 since it is already implied by (1) combined with (2).

In fact, we will use Theorem 1.2 to prove Theorem 1.3. Moreover, the equivalence of conditions (ii) and (iii) is immediate from [Fr00, Fu00a]. Since the inequalities of Theorem 1.3 are all homogeneous, the equivalence of (i) and (ii) immediately implies Theorem 1.1.

1.4. Further comparisons to the literature. The proof of the saturation theorem given in [KnTa99] is combinatorial, employing their *honeycomb model* for $c_{\lambda,\mu}^\nu$. In contrast, P. Belkale first geometrically proves the equivalence “(i) \Leftrightarrow (ii)” of Theorem 1.2, and then deduces the saturation theorem as an easy consequence [Be06]. By comparison, our main tool is again a new combinatorial model for $C_{\lambda,\mu}^\nu$; we similarly deduce (equivariant) saturation from the eigenvalue inequalities.

It seems plausible to give geometric proofs of our theorems, along the lines of [Be06], using the equivariant moving lemma of the first author [An07]. This approach is especially pertinent where one does not have good combinatorial control of the equivariant Schubert coefficients, e.g., in the case of minuscule G/P , cf. [PuSo09]. (P. Belkale and S. Kumar [BeKu06] also consider the vanishing problem for classical Schubert structure constants associated to more general G/P ’s.)

Schubert calculus on Grassmannians has two other basic extensions that have been extensively studied: quantum and K -theoretic Schubert calculus. It is therefore natural to ask if and how the Horn problem may extend in each of these directions. The first of these was studied by P. Belkale [Be08], who established a relationship between an eigenvalue problem for products of unitary matrices and analogues of the saturation and Horn theorems for quantum cohomology of Grassmannians. (A combined quantum-equivariant extension is plausible, and investigating this seems worthwhile, but we have not yet undertaken such an investigation.) For the second, let $k_{\lambda,\mu}^\nu$ denote the K -theoretic structure constant with respect to the basis structure sheaves of Schubert varieties. The “easy” implication $k_{\lambda,\mu}^\nu \neq 0 \implies k_{N\lambda,N\mu}^{N\nu} \neq 0$ is false in general. (However, it is not known if the converse is true.) For example, [Bu02, Section 7] notes that $k_{(1),(1)}^{(2,1)} = -1$ but $k_{(2),(2)}^{(4,2)} = 0$. One also can check that the same partitions give a counterexample for saturation in T -equivariant K -theory, as well. Moreover, consider structure constants $\tilde{k}_{\lambda,\mu}^\nu$ for the multiplication of the dual basis in K -theory. Using the rule of [ThYo10, Theorem 1.6], one checks that $\tilde{k}_{N\cdot(1),N\cdot(1)}^{N\cdot(2,1)}$ is nonzero for $N = 1, 2, 3$ but zero for $N = 4$.

Summarizing, this paper addresses the remaining basic extension of Schubert calculus where a complete analogue of the saturation theorem exists. The result linking Friedland’s problem to equivariant Schubert calculus gives further evidence towards the thesis that Schubert calculus is a natural perspective for Horn’s problem. That said, some room for clarification of this thesis remains: on one hand, the polynomials $C_{\lambda,\mu}^\nu$ also have representation theoretic interpretations [MoSa99]; on the other hand, saturation fails in K -theoretic Schubert calculus (in three forms). Finding deeper connections and explanations for these phenomena seems an interesting possibility for future work.

1.5. Organization. In Section 2, we give the proof of Theorem 1.3, assuming a fact about the equivariant coefficients (Proposition 2.1) that we use in our inductive proof. This in turn is proved in Section 3, after a review of the combinatorial rule of [ThYo12].

2. PROOF OF THEOREM 1.3

As remarked above, we only need to show that parts (i) and (ii) of Theorem 1.3 are equivalent. The basic idea is to run an induction on the degree $p = |\lambda| + |\mu| - |\nu|$, simultaneously with an induction on r . The key properties about $C_{\lambda,\mu}^\nu$ we need are the following:

Proposition 2.1. Assume $C_{\lambda,\mu}^\nu \neq 0$. Then:

- (I) $C_{\lambda,\mu^\uparrow}^\nu \neq 0$ for any $\mu \subset \mu^\uparrow \subseteq \nu$;
- (II) if $|\nu| < |\lambda| + |\mu|$, then there is a μ^\downarrow such that $|\nu| = |\lambda| + |\mu^\downarrow|$, with $\mu^\downarrow \subsetneq \mu$ and $C_{\lambda,\mu^\downarrow}^\nu \neq 0$.

We postpone the proof to the next section.

Proof of Theorem 1.3, (i) \Rightarrow (ii). If $C_{\lambda,\mu}^\nu \neq 0$, then by Proposition 2.1(II), we can find $\lambda^\downarrow \subseteq \lambda$ such that $|\lambda^\downarrow| + |\mu| = |\nu|$ and $C_{\lambda^\downarrow,\mu}^\nu = c_{\lambda^\downarrow,\mu}^\nu \neq 0$. By Theorem 1.2, for any triple (I, J, K) such that $c_{\lambda(I),\lambda(J)}^{\lambda(K)} \neq 0$, we have

$$\sum_{i \in I} \lambda_i^\downarrow + \sum_{j \in J} \mu_j \geq \sum_{k \in K} \nu_k.$$

Since $\sum_{i \in I} \lambda_i \geq \sum_{i \in I} \lambda_i^\downarrow$, (i) implies (ii), as desired. \square

Recall the bijection between d -subsets $I \subseteq [r]$ and partitions $\lambda = \lambda(I)$ in the $d \times (r - d)$ rectangle, and write $\sigma_I = \sigma_{\lambda(I)}$ for the corresponding Schubert class in the ordinary cohomology ring $H^*\text{Gr}_d(\mathbb{C}^r)$. Define $I^\vee \subseteq [r]$ as

$$I^\vee := \{r + 1 - i_d < \dots < r + 1 - i_2 < r + 1 - i_1\};$$

this is the subset associated to the shape λ^\vee which is defined by taking the complement of λ in $d \times (r - d)$ and rotating by 180 degrees.

We need an alternative characterization of Theorem 1.3(ii).

Lemma 2.2. Let λ, μ, ν be partitions as in Theorem 1.3. Then λ, μ, ν satisfy condition (ii) of Theorem 1.3 if and only if for any triple (I, J, K) such that $\sigma_I \sigma_J \sigma_{K^\vee} \neq 0$ in the ordinary cohomology ring $H^*\text{Gr}_d(\mathbb{C}^r)$, we have

$$\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j \geq \sum_{k \in K} \nu_k.$$

Proof. If $c_{\lambda(I),\lambda(J)}^{\lambda(K)} \neq 0$, then $\sigma_I \sigma_J \sigma_{K^\vee} \neq 0$, so the inequalities of the Lemma include those of Theorem 1.3(ii), which proves the “if” statement. For the “only if” statement, we first recall a well-known fact (with proof, for completeness):

Claim 2.3. If $\sigma_\alpha \sigma_\beta \sigma_\gamma \neq 0$ then there exists $\tilde{\gamma}$ such that $\tilde{\gamma}^\vee \subseteq \gamma$ and $c_{\alpha,\beta}^{\tilde{\gamma}} \neq 0$.

Proof of Claim 2.3: We proceed by induction on $\Delta = |\alpha| + |\beta| + |\gamma|$. If $\Delta = d(r - d)$, then we can take $\tilde{\gamma} = \gamma^\vee$. Otherwise, suppose $\Delta < d(r - d)$ and choose $\gamma^\downarrow \subseteq \gamma$ such that $|\gamma| = |\gamma^\downarrow| + 1$. By the Pieri rule, we have

$$\sigma_{(1)} \sigma_{\gamma^\downarrow} = \sigma_\gamma + \text{other terms.}$$

If $\sigma_\alpha \sigma_\beta \sigma_\gamma \neq 0$, then $\sigma_\alpha \sigma_\beta (\sigma_{\gamma^\downarrow} \sigma_{(1)}) \neq 0$ and hence $\sigma_\alpha \sigma_\beta \sigma_{\gamma^\downarrow} \neq 0$. By induction, we can choose $\tilde{\gamma}^\vee \subseteq \gamma^\downarrow$ such that $c_{\alpha,\beta}^{\tilde{\gamma}} \neq 0$. But we also have $\tilde{\gamma}^\vee \subseteq \gamma$, so the claim is proved. \square

Now, if $\sigma_I \sigma_J \sigma_{K^\vee} \neq 0$, then by the claim there exists K_0 such that $\lambda(K_0^\vee) \subseteq \lambda(K^\vee)$ and $c_{\lambda(I),\lambda(J)}^{\lambda(K_0)} \neq 0$. Thus, the condition of Theorem 1.2(ii) implies

$$\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j \geq \sum_{k \in K_0} \nu_k,$$

which in turn implies the inequality of the Lemma, since $\sum_{k \in K_0} \nu_k \geq \sum_{k \in K} \nu_k$. \square

Lemma 2.2 allows us to replace the inequalities of Theorem 1.3(ii) by a larger set of inequalities. That is, we can instead use inequalities corresponding to (I, J, K) from the sets

$$S_d^r := \{(I, J, K) \subseteq [r]^3 \mid |I| = |J| = |K| = d \text{ and } \sigma_I \sigma_J \sigma_{K^c} \neq 0 \text{ in } H^* \text{Gr}_d(\mathbb{C}^r)\},$$

where $d < r$. This larger class of inequalities allows us to perform an induction.

We will first need a result from [Fu00a]. Let $I = \{i_1 < i_2 < \dots < i_d\}$ be a subset of $[r]$ of cardinality d and let F be a subset of $[d]$ of cardinality x . Define

$$I_F := \{i_f \mid f \in F\}.$$

If F is a subset of $[r - d]$ of cardinality y , then define

$$I_F^+ := I \cup (I^c)_F$$

where I^c denotes the complement of I in $[r]$.

Proposition 2.4. ([Fu00a, Proposition 1]) *Let $(I, J, K) \in S_d^r$.*

- (1) *If $(F, G, H) \in S_x^d$, then $(I_F, J_G, K_H) \in S_x^r$.*
- (2) *If $(F, G, H) \in S_y^{r-d}$, then $(I_F^+, J_G^+, K_H^+) \in S_{d+y}^r$.*

For any partitions λ and α , define $\phi(\lambda, \alpha)$ to be the partition with parts

$$\lambda_1, \dots, \lambda_d, \alpha_1, \dots, \alpha_d$$

arranged in weakly decreasing order.

Lemma 2.5. *Let λ, μ, ν and α, β, γ be partitions such that $C_{\lambda, \mu}^\nu \neq 0$ and $C_{\alpha, \beta}^\gamma \neq 0$.*

Then $C_{\phi(\lambda, \alpha), \phi(\mu, \beta)}^{\phi(\nu, \gamma)} \neq 0$.

Proof. Since $C_{\lambda, \mu}^\nu \neq 0$ and $C_{\alpha, \beta}^\gamma \neq 0$, by Proposition 2.1(II), there exist partitions $\lambda^\downarrow \subseteq \lambda$ and $\beta^\downarrow \subseteq \beta$ such that $c_{\lambda^\downarrow, \mu}^\nu \neq 0$ and $c_{\alpha, \beta^\downarrow}^\gamma \neq 0$. By Theorem 1.2(iii), there exist Hermitian matrices $A_1 + B_1 = C_1$ and $A_2 + B_2 = C_2$ such that A_i, B_i, C_i have eigenvalues $\lambda^\downarrow, \mu, \nu$ and $\alpha, \beta^\downarrow, \gamma$ respectively. Then

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$$

are Hermitian matrices with eigenvalues $\phi(\lambda^\downarrow, \alpha), \phi(\mu, \beta^\downarrow), \phi(\nu, \gamma)$ respectively.

Hence $c_{\phi(\lambda^\downarrow, \alpha), \phi(\mu, \beta^\downarrow)}^{\phi(\nu, \gamma)} \neq 0$. We have

$$\phi(\lambda^\downarrow, \alpha) \subseteq \phi(\lambda, \alpha) \subseteq \phi(\nu, \gamma) \text{ and } \phi(\mu, \beta^\downarrow) \subseteq \phi(\mu, \beta) \subseteq \phi(\nu, \gamma).$$

Thus, by Proposition 2.1(I), we conclude $C_{\phi(\lambda, \alpha), \phi(\mu, \beta)}^{\phi(\nu, \gamma)} \neq 0$. □

Remark 2.6. *The converse of Lemma 2.5 does not hold. For example, we have the $c_{(3), (2, 1, 1)}^{(3, 2, 1, 1)} \neq 0$, but $c_{(3), (1)}^{(2, 1, 1)} = c_{\emptyset, (2, 1)}^{(3)} = 0$.* □

We will need the following:

Lemma 2.7. *Let μ, ν be partitions with at most r parts such that $\mu \subseteq \nu$, and let I, J be subsets of $[r]$ of cardinality d . If $\lambda(I) \subseteq \lambda(J)$, then $\mu_{I^c} \subseteq \nu_{J^c}$.*

Proof. Let $I = (i_1 < \dots < i_d)$ and $J = (j_1 < \dots < j_d)$. Since $\lambda(I) \subseteq \lambda(J)$, we have that $i_k \leq j_k$ for all $k \in [d]$. This implies that if

$$I^c = (i'_1 < \dots < i'_{r-d}) \quad \text{and} \quad J^c = (j'_1 < \dots < j'_{r-d}),$$

then $i'_k \geq j'_k$ for all $k \in [r-d]$. Now $\mu_{i'_k} \leq \nu_{j'_k}$, since $\mu \subseteq \nu$. \square

Proof of Theorem 1.3, (ii) \Rightarrow (i): Let $p := |\lambda| + |\mu| - |\nu|$. We prove the converse by a double induction on p and r . If $p = 0$, then (ii) implies (i) by Theorem 1.2. The second induction is on r . In particular, if $r = 1$, then $C_{\lambda, \mu}^\nu \neq 0$ if and only if $\lambda_1 + \mu_1 \geq \nu_1$. These are the base cases of our induction.

Now assume $p > 0$ and $r > 1$. Suppose that (λ, μ, ν) satisfies the inequalities $(I, J, K) \in S_d^r$ for all $d < r$. In order to reach a contradiction, suppose $C_{\lambda, \mu}^\nu = 0$.

Remove any box from λ , to obtain a subpartition $\lambda^\downarrow \subseteq \lambda$ with $|\lambda| = |\lambda^\downarrow| + 1$. By Proposition 2.1(I) and (3) we know $C_{\lambda^\downarrow, \mu}^\nu = 0$. By induction on p , since $|\lambda^\downarrow| + |\mu| - |\nu| = p - 1$, we can find an inequality $(I, J, K) \in S_d^r$ satisfied by (λ, μ, ν) but not by $(\lambda^\downarrow, \mu, \nu)$. It is therefore true that

$$(4) \quad \sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j = \sum_{k \in K} \nu_k.$$

Let $\lambda_I := (\lambda_{i_1} \geq \dots \geq \lambda_{i_d})$ and similarly define μ_J and ν_K . Note that equation (4) is the statement $|\lambda_I| + |\mu_J| = |\nu_K|$. By assumption, (λ, μ, ν) satisfies (I, J, K) . If $(F, G, H) \in S_x^d$ (for any $x < d$) then by Proposition 2.4(1), (λ, μ, ν) also satisfies $(I_F, I_G, K_H) \in S_x^r$. This is the same as saying that $(\lambda_I, \mu_J, \nu_K)$ satisfies the inequalities $(F, G, H) \in S_x^d$ for all $x < d$. Thus, by Theorem 1.2, $c_{\lambda_I, \mu_J}^{\nu_K} \neq 0$.

Consider the partitions $\lambda_{I^c}, \mu_{J^c}, \nu_{K^c}$, where I^c, J^c, K^c are the complements of I, J, K in $[r]$. By Claim 2.3, $\lambda(I) \subseteq \lambda(K)$, so by Lemma 2.7, λ_{I^c} is a subpartition of ν_{K^c} . Similarly, we have that μ_{J^c} is a subpartition of ν_{K^c} . Now, again using our assumption that (λ, μ, ν) satisfies the inequalities $(I, J, K) \in S_d^r$, it follows from Proposition 2.4(2) and equation (4) that $(\lambda_{I^c}, \mu_{J^c}, \nu_{K^c})$ satisfies the inequalities $(F, G, H) \in S_y^{r-d}$ for $y < r - d$. Since $r - d < r$, we have by induction that $C_{\lambda_{I^c}, \mu_{J^c}}^{\nu_{K^c}} \neq 0$.

Since $c_{\lambda_I, \mu_J}^{\nu_K} \neq 0$ and $C_{\lambda_{I^c}, \mu_{J^c}}^{\nu_{K^c}} \neq 0$, Lemma 2.5 implies that $C_{\lambda, \mu}^\nu \neq 0$, which contradicts our original assumption. This completes the proof. \square

3. PROOF OF PROPOSITION 2.1

3.1. A Littlewood-Richardson rule via edge-labeled tableaux. There are several combinatorial rules for computing the equivariant structure constants $C_{\lambda, \mu}^\nu$; see, e.g., [MoSa99] and [KnTa03] for some early ones (the latter being the first one to manifest the ‘‘Graham-positivity’’ of the polynomials $C_{\lambda, \mu}^\nu$). However, in order to prove Proposition 2.1 we use a more recent rule of [ThYo12].

Consider Young diagrams λ, μ, ν inside Λ , with $\lambda, \mu \subseteq \nu$. An **equivariant Young tableau** of shape ν/λ and content μ is a filling of the boxes of the skew shape ν/λ , and labeling of some of the edges, by integers $1, 2, \dots, |\mu|$, where 1 appears μ_1 times, 2 appears μ_2 times, etc. The tableau is **semistandard** if the box labels weakly increase along rows (left to right), and all labels strictly increase down columns. (A single edge may be labeled by a *set* of integers, without repeats; the smallest of them must be strictly greater

than the label of the box above, and the largest must be strictly less than the label of the box below.)

Example 3.1. Below is an equivariant semistandard Young tableau on $(4, 2, 2)/(2, 1)$ of content $(3, 3, 2)$.

		1	1
	1	2, 3	2
2	3		

Let $\text{EqSSYT}(\nu/\lambda)$ be the set of all equivariant semistandard Young tableaux of shape ν/λ . A tableau $T \in \text{EqSSYT}(\nu/\lambda)$ is **lattice** if, for every column c and every label ℓ , we have:

$$(\# \ell\text{'s weakly right of column } c) \geq (\#(\ell + 1)\text{'s weakly right of column } c).$$

Given a tableau $T \in \text{EqSSYT}(\nu/\lambda)$, a (box or edge) label ℓ is **too high** if it appears weakly above the upper edge of a box in row ℓ . In the above example, all edge labels are too high.

Each box in the $r \times (n - r)$ rectangle Λ has a **distance** from the lower-left box: Using matrix coordinates for a box $\mathbf{x} = (i, j)$, we define $\text{dist}(\mathbf{x}) = r + j - i$.

Now suppose an edge label ℓ lies on the bottom edge of a box \mathbf{x} in row r . Let $\rho_\ell(\mathbf{x})$ be the number of times ℓ appears as a (box or edge) label strictly to the right of \mathbf{x} . We define

$$(5) \quad \text{apfactor}(\ell, \mathbf{x}) = t_{\text{dist}(\mathbf{x})} - t_{\text{dist}(\mathbf{x})+r-\ell+1+\rho_\ell(\mathbf{x})}.$$

When the edge label is not too high, this is always of the form $t_i - t_j$, for $i < j$. (In particular, it is nonzero.) Finally, we define² the **weight** of $T \in \text{EqSSYT}$ by

$$(6) \quad \text{apwt}(T) = \prod \text{apfactor}(\ell, \mathbf{x}),$$

the product being over all edge labels ℓ .

We can now state a combinatorial rule for equivariant Schubert calculus.

Theorem 3.2 ([ThYo12, Theorem 3.1]). *We have $C_{\lambda, \mu}^\nu = \sum_T \text{apwt}(T)$, where the sum is over all $T \in \text{EqSSYT}(\nu/\lambda)$ of content μ that are lattice and have no label which is too high.*

This is a nonnegative rule: from the definition of the weights, complete cancellation is impossible, so the existence of one such T means that $C_{\lambda, \mu}^\nu$ is nonzero. In particular:

Corollary 3.3. *The coefficient $C_{\lambda, \mu}^\nu$ is nonzero if and only if there exists a tableau $T \in \text{EqSSYT}(\nu/\lambda)$ of content μ which is lattice and has no label which is too high.*

Example 3.4. Consider the following lattice and semistandard tableau:

		1	1
	1		
2	2		

The associated apwt is $(t_1 - t_2)(t_4 - t_6)(t_5 - t_6)$; hence $C_{(4,1),(3,2,1)}^{(4,2,2)} \neq 0$. □

²In [ThYo12], this is called the “*a priori* weight”, to distinguish it from a weight arising from a sliding algorithm; hence the prefix “ap”.

3.2. Proof of Proposition 2.1. Our arguments for (I) and (II) are combinatorial, and both are based on Corollary 3.3.

Proof of (I): Let T be a witnessing tableau for $C_{\lambda, \mu}^\nu \neq 0$. That is, T is an (equivariant) semistandard tableau of shape ν/λ that is lattice, has content μ , and has no label that is too high. If $\mu = \nu$ then the desired assertion is trivial. Otherwise, by induction we quickly reduce to the case that μ^\uparrow/μ is a single box. Suppose this additional box is a corner added to row ℓ of the shape of μ . Our goal is to construct T^\uparrow by adding a single edge label ℓ to T so that T^\uparrow witnesses $C_{\lambda, \mu^\uparrow}^\nu \neq 0$.

Procedure to obtain T^\uparrow : Find the leftmost column c that

- does not already have ℓ in the same column; and
- placing ℓ in that column as an edge label does not make that new ℓ too high.

Place ℓ in column c as an edge label. (This placement is uniquely determined.)

Grant for the moment that such a column c exists for us to insert ℓ into. (In Claim 3.6 below, we will show that this is in fact the case.) Because we are adding ℓ to an edge, the assumptions allow us to conclude that T^\uparrow is semistandard (since the horizontal semistandard condition is vacuous here).

Claim 3.5. T^\uparrow is lattice.

Proof. Suppose T^\uparrow is not lattice. Hence there is a column d with strictly more ℓ 's than $(\ell - 1)$'s in the region R consisting of columns weakly to the right. Notice that since T is lattice and we put an additional ℓ in column c , then column d must be weakly left of column c .

Before inserting the ℓ , the region R had an equal number of ℓ 's and $(\ell - 1)$'s. Since we could put an ℓ into column c and not be too high, we could put an edge label in each of the columns strictly left of column d , unless they all had ℓ 's in them. However, in that case, since T is lattice, those columns must each also contain $\ell - 1$. Thus, $\mu_{\ell-1} = \mu_\ell$. Hence we could not add a corner in row ℓ to obtain μ^\uparrow , a contradiction. \square

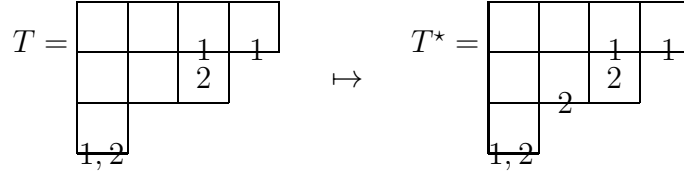
It remains to establish:

Claim 3.6. The column c exists.

Proof. To reach a contradiction, suppose otherwise. First assume there is a column d of T that does not have ℓ in it. We can take this column to be leftmost among all choices.

By our assumption for contradiction, we could not insert ℓ into column d because doing so would make it too high. Thus the edge e we would put ℓ into (as forced by semistandardness) is the upper edge of the box in row ℓ , or higher. If there is a box label m in the box of row ℓ in that column, then $m > \ell$ (by assumption). But then m was too high in T , a contradiction. Hence it must be the case that $\nu'_d \leq \ell - 1$ (where ν' is the transpose shape of ν). Since there were ℓ 's in each of the columns to the left, we conclude that the corner box μ^\uparrow/μ must be in column d or to the right. But $\mu'_d \leq \nu'_d \leq \ell - 1$, a contradiction, as we conclude that $\mu^\uparrow \not\subseteq \nu$. Finally, if column d does not exist, i.e., every column of T has an ℓ in it, then we conclude $\mu_\ell = \nu_\ell$ and again we see that $\mu^\uparrow \not\subseteq \nu$, a contradiction. \square

Example 3.7. We illustrate the procedure below:



Above, T witnesses $C_{(4,2,1),(3,2)}^{(4,3,1)} \neq 0$ and T^* witnesses $C_{(4,2,1),(3,3)}^{(4,3,1)} \neq 0$. \square

Proof of (II): As in the proof of (I), let T be a witnessing tableau for $C_{\lambda,\mu}^\nu \neq 0$. We will modify T to obtain T^* that witnesses $C_{\lambda,\mu^*}^\nu \neq 0$, where $\mu^* \subset \mu$ and $|\mu/\mu^*| = 1$. (Our T^* will have one fewer edge label than T .) The claim (II) follows by using this procedure to obtain a sequence of tableaux, each with one fewer edge label, until there are no edge labels.

Procedure to obtain T^* : Find the leftmost column $c^{(0)}$ of $T^{(0)} := T$ that has an edge label ℓ . If there are multiple edge labels in column $c^{(0)}$ let ℓ be the largest one. Define $T^{(1)}$ as $T^{(0)}$ with ℓ removed. If $T^{(1)}$ is lattice, set $T^* = T^{(1)}$ and stop. If $T^{(1)}$ is not lattice, there is a “bad column” weakly to the left of $c^{(0)}$, containing an offended $\ell + 1$, i.e., there are more $\ell + 1$ ’s weakly to its right than ℓ ’s; let $c^{(1)}$ be the rightmost of these bad columns. Define $T^{(2)}$ by replacing the offended $\ell + 1$ in column $c^{(1)}$ by ℓ . If $T^{(2)}$ is lattice, set $T^* = T^{(2)}$ and stop; otherwise, repeat by finding the rightmost column $c^{(2)}$ weakly left of $c^{(1)}$ containing an offended $\ell + 2$.

Continue: at step (i) , if $T^{(i)}$ is not lattice, find the column $c^{(i)}$ containing the rightmost offended label $\ell + i$ which is weakly left of column $c^{(i-1)}$. Define $T^{(i+1)}$ by replacing this $\ell + i$ by $\ell + i - 1$. When $T^{(i)}$ is lattice, or when $\ell + i$ is greater than all labels in $T^{(i)}$, stop and set $T^* = T^{(i)}$.

Let μ^* be the content of T^* . By construction, the shape of T^* is the same as that of T , namely ν/λ .

By our choice of starting point, all the labels that are changed to produce T^* (after ℓ is removed in the 0th step) are box labels.

Lemma 3.8. *The tableau T^* witnesses $C_{\lambda,\mu^*}^\nu \neq 0$. In particular,*

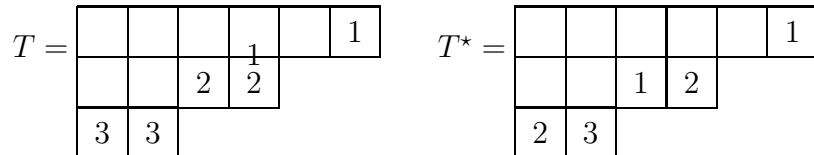
- (i) T^* is lattice.
- (ii) T^* is semistandard.
- (iii) No label of T^* is too high.

Hence the content μ^* of T^* is a partition, and T^* witnesses $C_{\lambda,\mu^*}^\nu \neq 0$.

(The final claim of the lemma is immediate from (i)-(iii) combined with Corollary 3.3. In particular, if μ^* is not a partition, then T^* cannot be lattice, contradicting (i).)

Before proving the lemma, we illustrate the procedure with a few examples.

Example 3.9. First consider:



Removing the edge label 1 leaves more 2's than 1's. Thus, replace the leftmost 2 by a 1. The result has more 3's than 2's. Conclude by replacing the leftmost 3 by a 2, leaving a lattice tableau T^* . \square

Example 3.10. Now,

$$T = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & 1 & 1 & 2 & 1 \\ \hline 2 & 2 & & & \\ \hline \end{array} \quad T^{(1)} = \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & 1 & 1 & 2 & \\ \hline 2 & 2 & & & \\ \hline \end{array}$$

Note that there are two offended labels 2 after the removal of $\ell = 1$. If the rightmost of the offended 2's is replaced by 1 (as demanded by the procedure) we obtain T^* while if the other offended 2 (in the first column) is replaced, we do not obtain a lattice tableau. \square

Example 3.11. Finally consider

$$T = \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & 1 & 1 \\ \hline 2 & 2 & & \\ \hline \end{array} \quad T^{(1)} = \begin{array}{|c|c|c|c|} \hline & & & 1 \\ \hline & & & 1 \\ \hline 2 & 2 & & \\ \hline \end{array}$$

Here there are two 2's left of $c^{(0)}$ but the rightmost of the two is *not* offended by the removal of $\ell = 1$, and the replacement of the 2 in the first column gives T^* . \square

We now turn to the proof of Lemma 3.8, proving parts (i) and (ii) by a simultaneous induction, and then considering part (iii).

Proof of (i) and (ii). We inductively argue, for $i \geq 1$, that $T^{(i)}$ is semistandard and $(\ell + i - 1)$ -**pseudolattice**. The latter means:

- (a) $T^{(i)}$ is lattice with respect to every pair of labels $k, k + 1$ for $k \neq \ell + i - 1$;
- (b) if it is not lattice with respect to $\ell + i - 1$ and $\ell + i$, any "bad column" b (containing an $\ell + i$, with fewer $(\ell + i - 1)$'s than $(\ell + i)$'s weakly to the right of it) satisfies the following: the number of $(\ell + i - 1)$'s weakly right of b is exactly one less than the number of $(\ell + i)$'s; and
- (c) any such bad column b is weakly left of column $c^{(i-1)}$.

Clearly (i) and (ii) follow from this induction, since by condition (a), $(\ell + i)$ -pseudolattice implies lattice whenever no label strictly greater than $\ell + i$ appears.

By assumption, $T = T^{(0)}$ is semistandard and lattice. Consider $T^{(1)}$. Since this was obtained by removing an ℓ , it remains lattice with respect to k and $k + 1$, for any k except possibly $k = \ell$. We see that $T^{(1)}$ is at least ℓ -pseudolattice, since $T^{(0)}$ must have had an equal number of ℓ 's and $(\ell + 1)$'s (weakly right of a bad column b) and we removed only one of the former. Also, $T^{(1)}$ is clearly semistandard.

Having dealt with the transition from $T^{(0)}$ to $T^{(1)}$, we henceforth assume $i \geq 1$.

Claim 3.12. Suppose $i \geq 1$. In column $c^{(i)}$ of $T^{(i)}$ there is an $\ell + i$ (by assumption), but there is no $\ell + i - 1$.

Proof. Otherwise we would conclude that there must be a column strictly to the right of $c^{(i)}$ (but weakly left of column $c^{(i-1)}$ in $T^{(i)}$) with an offended label $\ell + i$. \square

To obtain $T^{(i+1)}$, we replace the $\ell + i$ by $\ell + i - 1$. Locally, this replacement looks like:

$$T^{(i)} = \begin{array}{|c|c|c|} \hline a & b & c \\ \hline d & \ell + i & e \\ \hline f & g & h \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|} \hline a & b & c \\ \hline d & \ell + i - 1 & e \\ \hline f & g & h \\ \hline \end{array} = T^{(i+1)}.$$

(If the second column in the local diagram above is $c^{(0)}$, then there could be edge labels above the b or in the columns to the right. However, these labels do not affect any arguments below.)

Claim 3.13. $T^{(i+1)}$ is semistandard.

Proof. Clearly we have $\ell + i - 1 \leq e$ and $\ell + i - 1 < g$. That $b < \ell + i - 1$ is clear from the semistandardness of $T^{(i)}$ combined with Claim 3.12. It remains to show $d \leq \ell + i - 1$.

We have $d \leq \ell + i$, so in order to reach a contradiction, let us assume $d = \ell + i$. There are two cases. First, if $a = \ell + i - 1$, then by the semistandardness of $T^{(i)}$ we have $b = \ell + i - 1$, contradicting Claim 3.12. Second, if $a < \ell + i - 1$ (or if a does not exist), then it follows from Claim 3.12 that in $T^{(i-1)}$ the label $d = \ell + i$ witnesses that $T^{(i-1)}$ is not lattice with respect to $\ell + i - 1$ and $\ell + i$, contradicting our inductive hypothesis. Hence $d < \ell + i$, and $T^{(i+1)}$ is semistandard. \square

Claim 3.14. $T^{(i+1)}$ is $(\ell + i)$ -pseudolattice.

Proof. By induction, $T^{(i)}$ is $(\ell + i - 1)$ -pseudolattice. We assume it is not lattice; so to obtain $T^{(i+1)}$ we replaced an $\ell + i$ by an $\ell + i - 1$. Thus, clearly $T^{(i+1)}$ is lattice with respect to k and $k + 1$, for $k \notin \{\ell + i - 2, \ell + i - 1, \ell + i\}$.

$T^{(i+1)}$ is lattice with respect to $\ell + i - 1$ and $\ell + i$: Consider any bad column b of $T^{(i)}$ containing an offended label $\ell + i$, from the condition (b) defining pseudolatticeness. If $b = c^{(i)}$, then in $T^{(i+1)}$ the label $\ell + i$ has been replaced, so that column no longer has an offended $\ell + i$. For bad columns b further to the left, by (b) the deficit of $(\ell + i)$'s versus $(\ell + i - 1)$'s is exactly 1, and so the replacement of the $\ell + i$ in column $c^{(i)}$ by $\ell + i - 1$ eliminates this deficit (and in fact provides a surplus of one additional $\ell + i$ over the number of $(\ell + i - 1)$'s).

$T^{(i+1)}$ is lattice with respect to $\ell + i - 2$ and $\ell + i - 1$ (for $i \geq 2$): Although $T^{(i)}$ is $(\ell + i - 1)$ -pseudolattice and therefore lattice with respect to $\ell + i - 2$ and $\ell + i - 1$, in the transition from $T^{(i)}$ to $T^{(i+1)}$ we removed a box label $\ell + i$ from column $c^{(i)}$ and replaced it by $\ell + i - 1$. Potentially this might violate the lattice condition we are currently considering. Suppose there is such an offended label $\ell + i - 2$ in $T^{(i+1)}$, i.e., one with more $(\ell + i - 1)$'s in the columns weakly to the right than $(\ell + i - 2)$'s in the same region.

Where is this offended $\ell + i - 2$ in $T^{(i+1)}$? It cannot be in the region strictly right of column $c^{(i)}$ because $T^{(i)}$ and $T^{(i+1)}$ agree in that region and the former tableau is $\ell + i - 2, \ell + i - 1$ -lattice. Also, it cannot be in column $c^{(i)}$: That would mean that in $T^{(i)}$ there are an equal number of $(\ell + i - 2)$'s and $(\ell + i - 1)$'s weakly right of column $c^{(i)}$. However, since $c^{(i)}$ has an $\ell + i - 2$ in it, we conclude there are strictly more $(\ell + i - 1)$'s than $\ell + i - 2$ in the region strictly right of $c^{(i)}$ in $T^{(i)}$. That would mean $T^{(i)}$ is not lattice for these two labels, a contradiction.

Hence the offended $\ell + i - 2$ is strictly left of $c^{(i)}$ in $T^{(i+1)}$. However, notice that for this $\ell + i - 2$, the number of $(\ell + i - 2)$'s and $(\ell + i - 1)$'s weakly right of it in $T^{(i+1)}$ is the same as it is in $T^{(i-2)}$. (Only the placements of these labels differ, precisely in columns $c^{(i-2)}, c^{(i-1)}$)

and $c^{(i)}$.) Hence $\ell + i - 2$ would have been offended even in $T^{(i-2)}$, but $T^{(i-2)}$ is lattice with respect to $\ell + i - 2$ and $\ell + i - 1$, the final contradiction.

The lattice condition for $\ell + i$ and $\ell + i + 1$ in $T^{(i+1)}$: To check the condition (b), suppose we are given a bad column b in $T^{(i+1)}$. Since $T^{(i+1)}$ and $T^{(i)}$ are the identical strictly right of column $c^{(i)}$, and by induction $T^{(i)}$ is lattice with respect to $\ell + i$ and $\ell + i + 1$, we know the column b must be weakly left of $c^{(i)}$ (hence condition (c) holds). If $T^{(i+1)}$ is not lattice with respect to $\ell + i$ and $\ell + i + 1$, then in $T^{(i)}$ there were an equal number of $(\ell + i)$'s and $(\ell + i + 1)$'s weakly to the right of column b ; and now in $T^{(i+1)}$ there is one more of the former than the latter; this is condition (b).

We conclude that $T^{(i+1)}$ is $(\ell + i)$ -pseudolattice. □

Proof of (iii). We are given that no label of T is too high. Removing the initial edge label does not change this. Moreover, each step of our procedure only changes a box label to a label that is one smaller. If such a box label was not too high, replacing it by a smaller label will not change this either. □

This completes the proof of Lemma 3.8 and thus Proposition 2.1. □

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